

On an Extremal Problem of Fejér*

TAI-SHING LAU

Department of Statistics, SUNY at Buffalo, Amherst, New York 14261, USA

AND

W. J. STUDDEN

Department of Statistics, Purdue University, West Lafayette, Indiana 47907, USA

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The aim of this paper is to give new proofs to some theorems in Karlin and Studden's book [5] and Balázs's paper [2]. We also obtained some new results of a similar nature.

The general problem is as follows. Let $-\infty \leq a < b \leq \infty$ and let \mathcal{P}_{n+1} denote the class of $(n+1)$ -tuples (x_1, \dots, x_{n+1}) with $a \leq x_1 < \dots < x_{n+1} \leq b$ (x_1 and x_{n+1} are finite). Let $w(x)$ be a nonnegative function on $[a, b]$. Define

$$r_i(x) = \frac{l_i(x)}{w^{1/2}(x_i)}, \quad i = 1, 2, \dots, n+1,$$

where

$$l_i(x) = \frac{L_{n+1}(x)}{L_{n+1}(x_i)(x - x_i)} \quad \text{and} \quad L_{n+1}(x) = \prod_{i=1}^{n+1} (x - x_i)$$

are the well-known Lagrange polynomials associated with interpolation. The extremal problem is to determine the value

$$\begin{aligned} N &= \inf_{\mathcal{P}_{n+1}} \sup_{a \leq x \leq b} w(x) \{r_1^2(x) + \dots + r_{n+1}^2(x)\} \\ &= \inf_{\mathcal{P}_{n+1}} \tilde{N}(x_1, \dots, x_{n+1}) \end{aligned} \tag{1}$$

and a set of points $\{x_i\}_{i=1}^{n+1}$ which minimizes \tilde{N} for a fixed $w(x)$.

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Fejér [4] showed that, in the case $w(x) = 1$ and $[a, b] = [-1, 1]$, the quantity N is minimized if $\{x_i\}_{i=1}^{n+1}$ are the roots of the equation $(1 - x^2) P'_n(x) = 0$ where $P_n(x)$ is the n th Legendre polynomial and the minimum value of N is 1. The cases that will be treated in this paper are listed in the following:

- A. $[a, b] = [-1, 1]$.
 - (i) $w(x) = 1$
 - (ii) $w(x) = (1 - x)^{\alpha+1} (1 + x)^{\beta+1}$ ($\alpha > -1, \beta > -1$)
 - (iii) $w(x) = (1 - x)^{\alpha+1}$ ($\alpha > -1$)
 - (iv) $w(x) = (1 + x)^{\beta+1}$ ($\beta > -1$)
 - (v) $w(x) = (1 - x^2)^{\alpha+1} |x|^\gamma$ ($\alpha > -1, \gamma > 0, n$ is odd)
 - (vi) $w(x) = |x|^\gamma$ ($\gamma > 0, n$ is odd).
- B. $[a, b] = [0, \infty]$.
 - (i) $w(x) = e^{-x}$
 - (ii) $w(x) = x^{\alpha+1} e^{-x}$ ($\alpha > -1$).
- C. $[a, b] = [-\infty, \infty]$.
 - (i) $w(x) = e^{-x^2}$
 - (ii) $w(x) = |x|^\gamma e^{-x^2}$ ($\gamma > 0, n$ is odd).

Karlin and Studden [5] gave proofs for cases A(i), (ii); B(i), (ii); and C(ii). In addition Balázs [2] considered cases A(iii), (iv) and C(ii). Cases A(v), (vi) are new and were suggested by Askey (1). The solutions to the above cases are listed in Theorem 1 below. The following notations, which are mainly from Szegő [9], will be used:

- $P'_n(x)$ denotes the derivative of the n th Legendre polynomial.
- $P_n^{(\alpha, \beta)}(x)$ denotes the n th Jacobi polynomial.
- $R_n^{(\alpha, \gamma)}(x)$ denotes the n th orthogonal polynomial with respect to $(1 - x^2)^\alpha |x|^\gamma$.
- $L_n(x)$ denotes the n th Laguerre polynomial.
- $L_n^\alpha(x)$ denotes the n th Generalized Laguerre polynomial.
- $H_n(x)$ denotes the n th Hermite polynomial.
- $H_n^\gamma(x)$ denotes the n th Sonin–Markov polynomial (generalized Hermite polynomial) orthogonal with respect of $|x|^\gamma e^{-x^2}$.

THEOREM 1. *The solutions for cases A, B, and C are the zeros of the following polynomials:*

- A. $[a, b] = [-1, 1]$.
 - (i) $(1 - x^2) P'_n(x)$
 - (ii) $P_{n+1}^{(\alpha, \beta)}(x)$
 - (iii) $(1 + x) P_n^{(\alpha, 1)}(x)$

- (iv) $(1-x) P_n^{(1,\beta)}(x)$
 - (v) $R_{n+1}^{(\alpha,\gamma)}(x)$ (n is odd)
 - (vi) $(1-x^2) R_{n-1}^{(1,\gamma)}(x)$ (n is odd).
- B. $[a, b] = [0, \infty]$.
- (i) $xL_n(x)$
 - (ii) $L_{n+1}^\alpha(x)$.
- C. $[a, b] = [-\infty, \infty]$.
- (i) $H_{n+1}(x)$
 - (ii) $H_{n+1}^\gamma(x)$ (n is odd).

The value of N in each case is 1.

In this paper we give a new proof using the coefficients in certain continued fraction expansions of Stieltjes transforms or equivalently, the coefficients in the three-term recursion formula for arbitrary orthogonal polynomials with leading coefficient one. Part of the proofs is modeled after the results in Karlin and Studden [5] which use a theorem of Kiefer and Wolfowitz [6] from statistical design theory. The problem of identifying the points $\{x_i\}_{i=1}^{n+1}$ minimizing \tilde{N} in (1) is turned into one of identifying the polynomial coefficients maximizing certain determinants. The determinants in each case can be simply written down in terms of the coefficients from the continued fraction expansion and the maximization trivially carried out. The resulting coefficients are then identified with the solutions in Theorem 1. The solutions are greatly unified and all the cases for a given interval $[-1, 1]$, $[0, \infty]$, or $[-\infty, \infty]$ can be handled at the same time.

In the following we outline the proof in a number of steps. The details for the various steps are given later.

Step 1. Let ξ denote a probability measure with mass $1/(n+1)$ on each point x_i , $i=1, 2, \dots, n+1$. Write $f'(x) = (1, x, \dots, x^n)$ and $M(\xi) = \int f(x) f'(x) w(x) d\xi(x)$. Then

$$\begin{aligned} (n+1) w(x) \sum_{i=1}^{n+1} r_i^2(x) &= w(x) f'(x) M^{-1}(\xi) f(x) \\ &= v(x; \xi) \end{aligned} \quad (2)$$

and hence

$$N = \inf_{\xi} \sup_x v(x; \xi) \quad (3)$$

The infimum in (3) is over ξ with equal masses on $n+1$ points. It turns out in our situation that the same value is obtained for N if ξ is allowed to be an arbitrary probability measure.

Step 2. Let ξ be an arbitrary probability measure.

THEOREM 2 (Kiefer-Wolfowitz). *The conditions*

- (i) ξ^* minimizes $\sup v(x; \xi)$
- (ii) ξ^* maximizes $|M(\xi)| = \det M(\xi)$
- (iii) $\sup_x v(x; \xi^*) = n + 1$

are equivalent. The set B of all ξ^* satisfying these conditions is linear, and $M(\xi)$ is the same for all ξ^* in B .

Step 3. If ξ^* maximizes $|M(\xi)|$, where $w(x)$ is any one of the cases in A, B, or C, then ξ^* is supported on $n + 1$ points x_1, x_2, \dots, x_{n+1} and $\xi^*(x_i) = 1/(n + 1)$. For ξ supported on $n + 1$ points

$$|M(\xi)| = |M_0(\xi)| \sum_{i=1}^{n+1} w(x_i). \tag{4}$$

Here $M_0(\xi)$ denotes the matrix $M(\xi)$ corresponding to $w(x) \equiv 1$. Note that the weights for ξ in (4) will still be thought of as arbitrary. If $w(x)$ is defined on $[-\infty, \infty]$ or $[-1, 1]$ and is symmetric about 0, then ξ^* may be assumed symmetric also. Symmetric arguments are used only for case C. Any symmetric situations in case A result directly in a symmetric solution without a separate argument.

For the next step we introduce some parameters to characterize ξ . We then write down (4) in terms of these parameters, perform the maximization, and identify the solutions in Theorem 1.

The following theorem gives us a set of parameters to characterize a probability measure on the various intervals.

THEOREM 3. (A) *The Stieltjes transform of every probability measure on $[-1, 1]$ with $n + 1$ support points has the continued fraction expansion*

$$\int_{-1}^1 \frac{d\xi(x)}{z-x} = \frac{1}{z+1-2\zeta_1} - \frac{4\zeta_1\zeta_2}{z+1-2\zeta_2-2\zeta_3} - \frac{4\zeta_3\zeta_4}{z+1-2\zeta_3-2\zeta_4} - \dots - \frac{4\zeta_{2n-1}\zeta_{2n}}{z+1-2\zeta_{2n}-2\zeta_{2n+1}} \tag{5}$$

or equivalently

$$\int_{-1}^1 \frac{d\xi(x)}{z-x} = \frac{1}{z+1} - \frac{2\zeta_1}{1} - \frac{2\zeta_2}{z+1} - \dots - \frac{2\zeta_{2n}}{z+1} - \frac{2\zeta_{2n+1}}{1}, \tag{5a}$$

where $\zeta_1 = p_1$, $\zeta_i = q_{i-1} p_i$ for $i \geq 2$, $0 < p_i < 1$ for $i < 2n$, $0 \leq p_i \leq 1$ for $i = 2n, 2n + 1$, $q_i = 1 - p_i$ for all i .

(B) Every probability measure ξ on $[0, \infty]$ with $n + 1$ support points has a Stieltjes transform expansion

$$\int_0^\infty \frac{d\xi(x)}{z-x} = \frac{1}{z} - \frac{d_1}{1} - \frac{d_2}{z} - \dots - \frac{d_{2n}}{z} - \frac{d_{2n+1}}{1}, \tag{6}$$

where $d_i > 0$ for $i \leq 2n$ and $d_{2n+1} \geq 0$.

(C) Every probability measure ξ on $[-\infty, \infty]$ with $n + 1$ support points satisfies

$$\int_{-\infty}^\infty \frac{d\xi(x)}{z-x} = \frac{1}{z-b_1} - \frac{a_1}{z-b_2} - \frac{a_2}{z-b_3} - \dots - \frac{a_n}{z-b_{n+1}}, \tag{7}$$

where $a_i > 0$ for $i \leq n$ and $-\infty < b_i < \infty$ for $i \leq n + 1$. For symmetric ξ the b_i are all zero.

Proof. For part A, see Wall [11 or 12]. The form given in (5) is a contraction of (5a). Parts B and C follow from Shohat and Tamarkin [8, p. 47 and 32, respectively].

In order to calculate the determinant $|M(\xi)|$ in (4) we need both the determinant $|M_0(\xi)|$ and the product $\prod_{i=1}^{n+1} w(x_i)$.

Step 4. The determinants $|M_0(\xi)|$ for the three cases are given by

- (A) $|M_0(\xi)| = \prod_{i=1}^n (\zeta_{2i-1} \zeta_{2i})^{n-i+1}$
- (B) $|M_0(\xi)| = \prod_{i=1}^n (d_{2i-1} d_{2i})^{n-i+1}$
- (C) $|M_0(\xi)| = \prod_{i=1}^n a_i^{n-i+1}$.

Step 5. The products $\prod_{i=1}^{n+1} w(x_i)$ for the three cases are as follows:

(A) In this case let us write $w(x) = (1-x)^{\alpha+1} (1+x)^{\beta+1} |x|^\gamma$ with the understanding that $\gamma > 0$ iff $\alpha = \beta$. We have

$$\prod w(x_i) = \text{const} \left(\prod_{i=1}^{2n+1} q_i \right)^{\alpha+1} \left(\prod_{i=0}^n \zeta_{2i+1} \right)^{\beta+1} (p_2 q_4 p_6 q_8 \cdots q_{2n-2} p_{2n})^\gamma.$$

$$\begin{aligned} \text{(B)} \quad \prod w(x_i) &= \prod_{i=1}^{n+1} x_i^{\alpha+1} \exp\left(-\sum_{i=1}^{n+1} x_i\right) \\ &= \left(\prod_{i=0}^n d_{2i+1}\right)^{\alpha+1} \exp\left(-\sum_{i=1}^{2n+1} d_i\right). \end{aligned}$$

(C) Here we note that $\gamma > 0$ iff n is odd,

$$\prod w(x_i) = \prod_{i=1}^{n+1} |x_i|^\gamma \exp\left(-\sum_{i=1}^{n+1} x_i^2\right) \\ = (a_1 a_3 \cdots a_n)^\gamma \exp\left(-2 \sum_{i=1}^n a_i\right).$$

Step 6. If we now multiply the corresponding expressions for $|M(\xi_0)|$ and $\prod w(x_i)$ from Steps 4 and 5 in each case, the resulting value for $|M(\xi)|$ can be maximized using simple calculus. The resulting parameters in each case are given by

(A)
$$p_{2i+1} = \frac{\beta + n - i + 1}{\alpha + \beta + 2(n - i + 1)}, \quad 0 \leq i \leq n$$

$$p_{2i} = \frac{\gamma + n - i + 1}{\alpha + \beta + \gamma + 2(n - i + 1) + 1}, \quad i \text{ odd}, 1 \leq i \leq n$$

$$p_{2i} = \frac{n - i + 1}{\alpha + \beta + \gamma + 2(n - i + 1) + 1}, \quad i \text{ even}, 1 \leq i \leq n.$$

(B)
$$d_{2i+1} = \alpha + n - i + 1, \quad 0 \leq i \leq n$$

$$d_{2i} = n - i + 1, \quad 1 \leq i \leq n.$$

(C)
$$a_{2i+1} = \frac{n - 2i + \gamma}{2}, \quad 0 \leq i \leq \frac{n-1}{2}, \quad n \text{ odd}, \gamma > 0$$

$$a_{2i} = \frac{n - 2i + 1}{2}, \quad 1 \leq i \leq \frac{n-1}{2}, \quad n \text{ odd}, \gamma > 0$$

$$a_i = \frac{n - i + 1}{2}, \quad 1 \leq i \leq n, \quad \gamma = 0.$$

Step 7. Identify the parameters in Step 6 to obtain the results in Theorem 1.

Proofs of Steps 1-7

Step 1. Equation (2) follows by noting that $v(x; \xi)$ is invariant under basis change for the powers $1, x, x^2, \dots, x^n$ and we convert to the lagrange form $l_1(x), \dots, l_{n+1}(x)$.

Step 2. This is the Kiefer-Wolfowitz "equivalence theorem" from statistical design theory. The proof is actually fairly simple and uses the fact that $\ln |M(\xi)|$ is strictly concave in M . Thus a local maximum is a global maximum and the maximizing ξ^* have the same M value. Let

By expanding the above determinants for cases (B) and (C), we obtain

$$(B) \quad z^{n+1} - \left(\sum_{i=1}^{2n+1} di \right) z^n + \dots + (-1)^{n+1} d_1 d_3 \dots d_{2n+1}$$

$$(C) \quad z^{n+1} - \left(\sum_{i=1}^n a_i \right) z^n + \dots, \quad n \text{ is even}$$

$$z^{n+1} - \left(\sum_{i=1}^n a_i \right) z^n + \dots + (-1)^{(n+1)/2} a_1 a_3 \dots a_n \quad n \text{ is odd.}$$

The term $\prod_{i=1}^{n+1} w(x_i)$ can be expressed in terms of d_i 's and a_i 's for cases B and C, respectively. More explicitly, we have

$$(B) \quad e^{-\sum_{i=1}^{n+1} x_i} \prod_{i=1}^{n+1} x_i^{\alpha+1} = e^{-\sum_{i=1}^{n+1} di} d_1 d_3 \dots d_{2n+1}^{\alpha+1}$$

$$(C) \quad e^{-\sum_{i=1}^{n+1} x_i^2} \prod_{i=1}^{n+1} |x_i|^\gamma = e^{-2\sum_{i=1}^n a_i} (a_1 a_2, \dots, a_n)^\gamma \quad n \text{ is odd and } \gamma > 0.$$

$$e^{-\sum_{i=1}^{n+1} x_i^2} = e^{-2\sum_{i=1}^n a_i} \quad n \text{ arbitrary } \gamma = 0.$$

To compute $\prod_{i=1}^{n+1} w(x_i)$ for case A, we first show by induction,

$$D_{n+1}(1) = \prod_{i=1}^{n+1} (1 - x_i) = \text{const} \prod_{i=1}^{2n+1} q_i$$

$$D_{n+1}(-1) = \prod_{i=1}^{n+1} (1 + x_i) = \text{const} \prod_{i=0}^n \zeta_{2i+1}.$$

Next, it can be seen that $1 - 2\zeta_1 = 0$ and $1 - 2\zeta_{2k} - 2\zeta_{2k+1} = 0$ for $k \geq 1$ if ξ is symmetric about 0. This is the case iff $P_i = \frac{1}{2}$ for all i . (This is the same as $b_i = 0$ in the expression (7).) Thus, in case ξ is symmetric about 0 and n is odd, we have

$$D_{n+1}(0) = \prod_{i=1}^{n+1} x_i = \text{const} \cdot p_2 q_4 p_6 q_8 \dots q_{2n-2} p_{2n}.$$

Writing $w(x) = (1 - x)^{\alpha+1} (1 + x)^{\beta+1} |x|^\alpha$ with the understanding that $\alpha \neq 0$ iff $\alpha = \beta$. We have

$$\prod_{i=1}^{n+1} w(x_i) = \text{const} \left(\prod_{i=1}^{2n+1} q_i \right)^{\alpha+1} \left(\prod_{i=0}^n \zeta_{2i+1} \right)^{\beta+1} (p_2 q_4 p_6 q_8 \dots q_{2n-2} p_{2n})^\gamma.$$

Step 6. This step is straightforward. We multiply the values for $\prod w(x_i)$ and $|M(\xi_0)|$, for cases A, B, and C separately, to give $|M(\xi)|$. The maximization in each case is relatively easy.

Step 7. The identification of the appropriate roots in Theorem 1 from the parameters in Step 6 revolves around an interesting symmetry property of the parameters defined generally in Theorem 3. This property is stated in the following theorem. The proof uses an induction argument and will be omitted. Full details can be found in Lau [7].

THEOREM 4. *If $0 < p_i < 1$ for $i = 1, 2, \dots, m$ then*

(a) *the probability measures corresponding to the sequences $(p_1, p_2, \dots, p_m, 0)$ and $(p_m, p_{m-1}, \dots, p_1, 0)$ have the same support. Further, the probability measures corresponding to $(p_1, p_2, \dots, p_m, 1)$ and $(q_m, q_{m-1}, \dots, q_1, 1)$ have the same support. Similarly in cases (B) and (C), if $d_i > 0, a_i > 0$ for $i = 1, \dots, m$ then*

(b) *$(d_1, \dots, d_m, 0)$ and $(d_m, \dots, d_1, 0)$ have the same support and*

(c) *$(a_1, \dots, a_m, 0)$ and $(a_m, \dots, a_1, 0)$ have the same support.*

With the aid of Theorem 4 the results given in Theorem 1 are more or less immediate. The required Stieltjes expansions for the identification for cases A, B, and C are taken from Van Rensselaer [10, case A on pages 51 and 56, case B on page 41, and case C on page 45]. Some of these are also given in Wall [12, formulas (89.16) and (92.4)].

To identify parts (i)–(iv) in case A we make use of the fact that the Stieltjes transform of the Jacobi weight function $(1-x)^{\alpha+1}(1+x)^{\beta+1}$ has an infinite expansion as in case A in Theorem 3 with parameters given by

$$\begin{aligned}
 p_{2k} &= \frac{k}{\alpha + \beta + 2k + 3}, & k \geq 1, \\
 p_{2k+1} &= \frac{\beta + k + 2}{\alpha + \beta + 2k + 4}, & k \geq 0.
 \end{aligned}
 \tag{8}$$

Special cases of interest correspond to $\alpha = \beta = -\frac{3}{2}$ and $\alpha = \beta = -1$. The situation $\alpha = \beta = -\frac{3}{2}$ is associated with the Chebyshev polynomials of the 1st kind and we have $p_i = \frac{1}{2}$, for all i . For the Lebesgue or uniform measure with $\alpha = \beta = -1$ the resulting parameters are $p_i = \frac{1}{2}$ for i odd and $p_{2i} = i/(2i + 1)$.

Consider part (i) of case A. The parameters p_i maximizing $|M(\xi)|$ are given from Step 6 as

$$\begin{aligned}
 p_{2i+1} &= \frac{1}{2} & i = 0, 1, \dots, n-1 \\
 p_{2i} &= \frac{n-i+1}{2(n-i+1)+1} & i = 1, 2, \dots, n-1 \\
 p_{2n} &= 1.
 \end{aligned}
 \tag{9}$$

On comparing this sequence with the sequence for the uniform measure we note they both have $p_i = \frac{1}{2}$ for i odd. The even indexed parameters for the uniform measure are $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \dots$, etc. while the even parameters, maximizing $|M(\xi)|$ are given, in reverse order, by $\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \dots$.

Part (a) of Theorem 4 implies that the measure maximizing $|M(\xi)|$ for part (i), case A has the same support as the sequence

$$\begin{aligned} p_{2i+1} &= \frac{1}{2}, & i &= 0, 1, \dots, n-1, \\ p_{2i} &= i/(2i+1), & i &= 1, 2, \dots, n-1, \\ p_{2n} &= 1. \end{aligned} \tag{10}$$

This sequence is obtained by “truncating” the sequence from the uniform measure by setting $p_{2n} = 1$. The resulting support in this case is on ± 1 and the zeros of the $(n-1)$ st polynomial orthogonal with respect to $(1-x^2) dx$ which is precisely the polynomial $P'_n(x)$ given in part (i), case A of Theorem 1.

It is interesting to note that the finitely supported measure corresponding to (10) is associated with a classical Gauss-type quadrature formula using the end points ± 1 . The reversed sequence in (1) has precisely the same support and uniform weights.

The same phenomenon occurs in all the other cases including cases B and C. To illustrate further, consider part (ii) of case A. Here the maximizing sequence from Step 6 is given by

$$\begin{aligned} p_{2i+1} &= \frac{\beta + n + 1 - i}{\alpha + \beta + 2n - 2i + 2}, & 0 \leq i \leq n, \\ p_{2i} &= \frac{n + 1 - i}{\alpha + \beta + 2n - 2i + 3}, & 1 \leq i \leq n, \\ p_{2n+2} &= 0. \end{aligned} \tag{11}$$

If we take the sequence from (8) with the weight $(1-x)^\alpha (1+x)^\beta$, truncate the sequence with $p_{2n+2} = 0$ and reverse the 1st $2n+1$ parameters we obtain (11). Using part (a) of Theorem 4 the support is on the zeros of Jacobi polynomial as stated in Theorem 1.

The remaining cases are similar and are omitted.

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